6. $xy''_{xx} + ay'_x + by = 0.$

1°. The solution is expressed in terms of the Bessel functions and modified Bessel functions:

$$y = \begin{cases} x^{\frac{1-a}{2}}\left[C_1 J_\nu\left(2\sqrt{bx}\,\right) + C_2 Y_\nu\left(2\sqrt{bx}\,\right)\right] & \text{if } bx > 0, \\ x^{\frac{1-a}{2}}\left[C_1 I_\nu\left(2\sqrt{|bx|}\,\right) + C_2 K_\nu\left(2\sqrt{|bx|}\,\right)\right] & \text{if } bx < 0, \end{cases}$$

where $\nu = |1 - a|$.

2°. For $a = \frac{1}{2}(2n + 1)$, where $n = 0, 1, \ldots$, the solution is:

$$y = \begin{cases} C_1 \dfrac{d^n}{dx^n} \cos\sqrt{4bx} + C_2 \dfrac{d^n}{dx^n} \sin\sqrt{4bx} & \text{if } bx > 0, \\ C_1 \dfrac{d^n}{dx^n} \cosh\sqrt{4|bx|} + C_2 \dfrac{d^n}{dx^n} \sinh\sqrt{4|bx|} & \text{if } bx < 0. \end{cases}$$

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